FOMO 2026 Lensing Problems

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1. **Reducing the Kinetic Energy Spread through Lensing.** The lowest effective temperatures (corresponding to the rms spread in kinetic energy), in the pK range, have been achieved by applying the analog of a lens to ultracold atoms. This technique is sometimes called matter wave lensing or delta-kick collimation/cooling. We will analyze how this method works.

   For simplicity, we will carry out analysis in a single dimension, although our treatment can readily be generalized to multiple dimensions. Imagine an ensemble of atoms with rms velocity spread Δv all located at position $x = 0$. We note that this is unrealistic both practically and also fundamentally (due to the Heisenberg uncertainty principle), but we will not worry about this until a later problem. The atoms are allowed to freely expand for a time T, and subsequently a harmonic potential of the form $V(x) = \frac{1}{2}kx^2$ is pulsed on for a time δt. In practice, such a potential can be approximately realized optically or magnetically, for example. An optical implementation would involve the dipole force of a laser field with an appropriate spatial profile. We assume that δt is sufficiently small that the atoms move a negligible distance during this time interval. Show that for an appropriate choice of δt, the velocity spread and therefore the effective temperature of the atom ensemble can be reduced to zero. What is this value of δt? Treat the trajectory of each atom classically.

   This process is analogous to using a lens to collimate light that has expanded from a perfect point source. An ideal lens imprints a quadratic phase profile on light that passes through, effectively playing the role of the quadratic potential.

2. **Lensing for the Case of Finite Initial Ensemble Size.** Realistically, the initial atom ensemble will always have a finite width in position. Let us denote the initial rms position spread as Δx_0. We let Δv_0 denote the initial velocity spread, so that the initial effective temperature is defined such that

   $$\frac{1}{2}k_B T_0 = \frac{1}{2}m\Delta v_0^2.$$

   Assume that the initial position and velocity distributions are uncorrelated.

   (a) Let us denote the final position and velocity spreads after lensing as Δx_f and Δv_f, respectively. The final temperature then satisfies $\frac{1}{2}k_B T_f = \frac{1}{2}m\Delta v_f^2$. For the lens applied after a free expansion time T, find the duration δt that minimizes the final temperature and show that this reduces to your result from problem 1 in the point source limit ($\Delta v_0 T \gg \Delta x_0$). Find corresponding minimum final temperature T_f in terms of T_0, Δx_0, and Δx_f. You can once again approximate each atom as following a classical trajectory.

(b) In classical mechanics, the enclosed phase space area of a system evolving under a conservative potential is conserved. Show that your result from part (a) is consistent with the conservation of phase space area.

3. **Quantum Mechanics in Phase Space: Introduction to the Wigner Function.** In analyzing lensing, we have seen in the first two problems that it is useful to consider trajectories in phase space. However, so far we have only considered classical phase space evolution, whereas a quantum mechanical wave packet experiencing a lensing sequence will undergo wavelike evolution. It turns out that under certain conditions, the approach taken in the first two problems of tracing classical trajectories can provide valid results. By analogy, although we know that in optics light obeys an underlying wave equation, ray tracing (analogous to our classical trajectory tracing here) can nevertheless provide an accurate approximation of how an optical field evolves as it passes through a system of lenses in many circumstances.

To understand the circumstances under which a quantum system exhibits this sort of semiclassical behavior, it is useful to introduce the Wigner function, which provides a quantum mechanical version of a phase space distribution function. For a density matrix $\hat{\rho}$, the corresponding Wigner function is defined as

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy\, e^{-ipy/\hbar} \left\langle x + \tfrac{1}{2}y \,\middle|\, \hat{\rho} \,\middle|\, x - \tfrac{1}{2}y \right\rangle \tag{1}$$

For the case of a pure state so that $\hat{\rho} = |\psi\rangle\langle\psi|$,

$$\begin{aligned} W(x,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy\, e^{-ipy/\hbar} \left\langle x + \tfrac{1}{2}y \,\middle|\, \psi \right\rangle \left\langle \psi \,\middle|\, x - \tfrac{1}{2}y \right\rangle \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy\, e^{-ipy/\hbar} \psi^*\left(x - \tfrac{1}{2}y\right) \psi\left(x + \tfrac{1}{2}y\right) \end{aligned} \tag{2}$$

In this problem, we will study some properties of the Wigner function.

(a) Show that integrating the Wigner function over all momenta p yields the probability density $P(x) \equiv \langle x|\, \hat{\rho}\, |x\rangle$ in x and that integrating the Wigner function over all positions x yields the probability density $P(p) \equiv \langle p|\, \hat{\rho}\, |p\rangle$ in p:

$$\int_{-\infty}^{\infty} dp\, W(x,p) = P(x) \tag{3}$$

$$\int_{-\infty}^{\infty} dx\, W(x,p) = P(p) \tag{4}$$

(b) Is $W(x,p)$ always real-valued? If so, prove this. If not, provide a counterexample.

(c) Consider an operator $\hat{A}$. The Weyl transform of this operator, which converts the operator to a complex-valued function of x and p, is defined as:

$$A_W(x,p) = \int_{-\infty}^{\infty} dy\, e^{-ipy/\hbar} \left\langle x + \tfrac{1}{2}y \,\middle|\, \hat{A} \,\middle|\, x - \tfrac{1}{2}y \right\rangle \tag{5}$$

Note that up to an overall factor, the Wigner function $W(x,p)$ is the Weyl transform of the density matrix $\hat{\rho}$. Show that the expectation value of $\hat{A}$ can be written as:

$$\left\langle \hat{A} \right\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, A_W(x,p)\, W(x,p) \tag{6}$$

(d) Find the Weyl transform of an operator $F(\hat{p})$ that is an arbitrary function of $\hat{p}$ and an operator $V(\hat{x})$ that is an arbitrary function of $\hat{x}$.

4. **Quantum-Classical Correspondence as Illuminated by the Wigner Function.** In classical mechanics, the time evolution of a phase space distribution $F(x,p,t)$ evolving under a Hamiltonian $H = \frac{p^2}{2m} + V(x)$ is described by the classical Liouville equation:

$$\frac{\partial F(x,p,t)}{\partial t} = -\frac{p}{m}\frac{\partial F(x,p,t)}{\partial x} + \frac{\partial V(x)}{\partial x}\frac{\partial F(x,p,t)}{\partial p} \tag{7}$$

(a) Where a point (x_0, p_0) at time $t = 0$ in phase space classically evolves at time t to position $x_c(x_0,p_0,t)$ and momentum $p_c(x_0,p_0,t)$, show that solutions to the classical Liouville equation have the form:

$$F(x,p,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_0\, dp_0\, \delta\big(x - x_c(x_0,p_0,t)\big)\,\delta\big(p - p_c(x_0,p_0,t)\big)\, F(x_0,p_0,0) \tag{8}$$

This solution corresponds to different points in phase space evolving according to the classical equations of motion.

(b) For a Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ where $V(\hat{x})$ is at most quadratic in $\hat{x}$ (i.e., no $\hat{x}^3$ terms or higher order terms), show that the time evolution of the Wigner function follows the classical Liouville equation. For our analysis of lensing, this means that considering the evolution of classical phase space trajectories to determine the time-evolution of the distributions of position and momentum is indeed a valid approach. (Hint: For the general case of a mixed state, you may find it helpful to express the density matrix as $\hat{\rho} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$ where the $\left|\psi_i\right\rangle$ are the different states in the mixture and p_i is the probability of the system being in state $\left|\psi_i\right\rangle$.)

(c) For general $V(\hat{x})$, there is a correction term to the classical Liouville equation that arises from nonzero third or higher order derivatives of $V(\hat{x})$. Show that in the general case, the Wigner function evolves according to:

$$\begin{aligned}\frac{\partial W(x,p,t)}{\partial t} = &-\frac{p}{m}\frac{\partial W(x,p,t)}{\partial x} + \frac{\partial V(x)}{\partial x}\frac{\partial W(x,p,t)}{\partial p} \\ &+ \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j+1)!}\left(\frac{\hbar}{2}\right)^{2j} \frac{\partial^{2j+1} V(x)}{\partial x^{2j+1}}\frac{\partial^{2j+1} W(x,p,t)}{\partial p^{2j+1}}\end{aligned} \tag{9}$$

This is known as the quantum Liouville equation. In some cases, even if there are nonzero higher order derivatives of $V(\hat{x})$, these derivatives are sufficiently small that the correction term can be ignored and the classical Liouville equation remains a useful approximation.

5. **Lensing with a Gaussian Laser Beam.** In practice, it is not possible to create a perfect quadratic potential. A common technique is to use the spatially varying AC Stark shift potential (also referred to as the dipole potential) from a Gaussian beam as the lensing potential. This leads to a potential $V(x) \sim \exp\left[-2x^2/w^2\right]$ where w is the width of the beam. For $x \ll w$, this potential is approximately quadratic. However, it significantly deviates from quadratic once x becomes comparable to w. Explain qualitatively why this Gaussian potential will not be able to achieve as low effective temperatures as a perfect quadratic potential.