Quantum Optimal Control in Ultracold Atomic Sensors: A Tutorial

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An outline of the talk

- The necessary background:
  - The creation, care, and feeding of cold and ultracold atoms
  - A dictionary of quantum optimal control
- A case study: sensing inertial motion with a shaken lattice interferometer
  - What even is atom interferometry?
  - Practical quantum control: what can you learn from my mistakes?

A dictionary of quantum control

A dictionary of quantum control

- Quantum control is, as the name says, the control of quantum systems.
- Two types:
  - *Coherent control* makes use of quantum interference to find the path to the desired state or process (the “physics first” approach)
  - *Optimal control* optimizes a quantum process and extracts physical insights/mechanisms (the “control first” approach)
- You can also control:
  - *State transfer* takes you from one specific quantum state $|\psi_A\rangle$ to another state $|\psi_B\rangle$
  - *Unitary operations* act on any quantum state in the same way $|\phi\rangle = U|\psi\rangle$
- Choose between two loops:
  - *Open-loop control* means that you are optimizing a model
  - *Closed-loop control* means that you are optimizing an experimental system
- Controls can be optimized for speed, sensitivity, robustness, etc, etc.

The state transfer problem

- State transfer: given an initial state $|\psi_0\rangle$ and some Hamiltonian $H\big(t, u(t)\big)$ that depends on time and some controls $u(t)$, can you reach a desired state $|\psi_d\rangle$ in some fixed time T?
- Final state after evolution

$$\left|\psi_f\right\rangle = U\big(0, T; u(t)\big)|\psi_0\rangle,$$

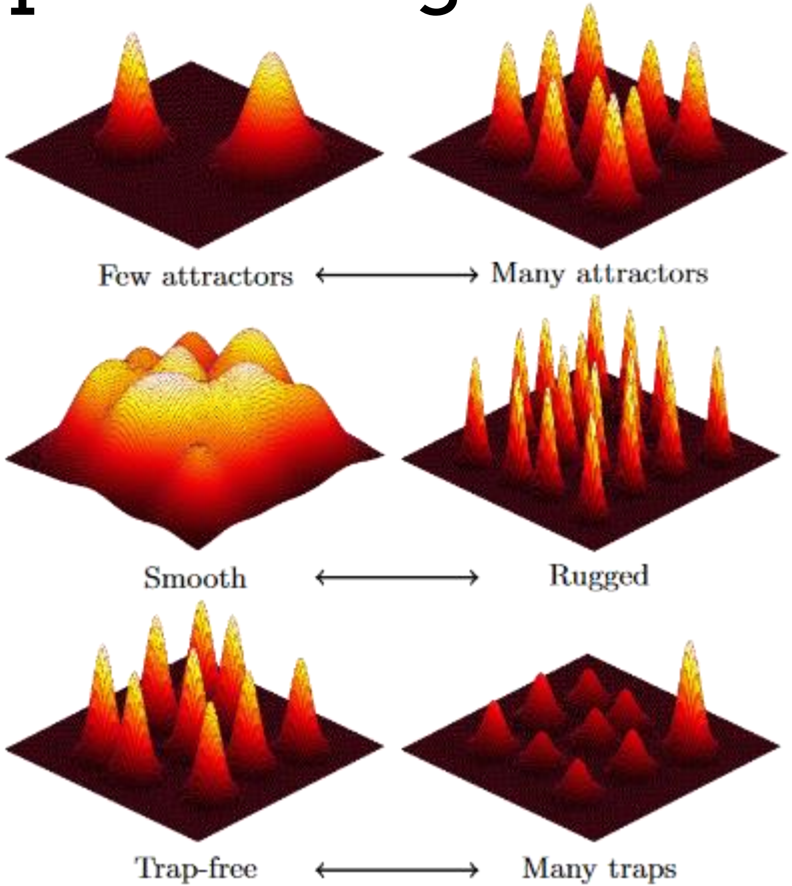
  where

$$U(0, T) = \boldsymbol{T}\mathrm{exp}\left\{\frac{\mathrm{i}}{\hbar}\int_0^T dt'\, H\big(t', u(t')\big)\right\}$$

- Usually $H(t, u(t)) = H_0 + \sum_n u_n(t) H_n$

Terminology on landscapes and algorithms

- Landscapes:
  - Typically, the set of optimization parameters defines a landscape. A landscape with more than one local minimum has local traps.
  - Your initial guess defines a *seed* in this landscape
  - Landscapes can be *rugged* in that small changes in controls cause large changes in the landscape, or *smooth* in the opposite case
  - It is said (Rabitz, 2004) that an *unconstrained* control landscape has no local maxima
- Algorithms:
  - An optimizer is *local* if it just climbs the nearest hill (e.g. gradient ascent), *global* if it can sample multiple hills
  - One can also define an algorithm's ability to explore a landscape vs. exploiting and focusing in on a local maximum
  - A *gradient-based* method requires the calculation of derivatives, a *gradient-free* method does not



The ever-important functional

- Every optimization problem needs a *functional*! You must decompose your problem into one number to be minimized (or maximized)!
- An example functional for a control $u(t)$, a final state $|\psi(T)\rangle$ and a desired state $|\psi_d\rangle$:

$$J = \lambda_0(1 - |\langle\psi_d|\psi(T)\rangle|^2) + \int_0^T dt\, \lambda_a(t)u(t)^2 + \int_0^T dt\, \lambda_b \dot{u}(t)^2 + \int_0^T dt\, \lambda_f \sum_n |\langle\phi_k|\psi(t)\rangle|^2$$

- Your functional can contain as many terms as you want but try to keep it as simple as possible.
- The weights (λ_i) determine the relative importance of each term.
- To optimize multiple state-to-state transfers with the same control field, sum multiple fidelity terms.
- Note that this functional picks a final time T
- The fastest possible transfer defines the *quantum speed limit*, which is difficult to prove rigorously for more than the simplest problems.

Fidelities for density matrices

- What about the case of *mixed states* that cannot be represented by bras and kets, or even density matrices in general?
- We could try $F = \mathrm{Tr}(\rho\sigma)$, but this doesn't work for mixed states…
- We have the Jozsa *fidelity axioms* [1] for four states ρ, σ, τ, ν:
  - $0 \leq F(\rho, \sigma) \leq 1$, and $F = 1$ only if $\rho = \sigma$
  - $F(\rho, \sigma) = F(\sigma, \rho)$
  - If $\rho = |\psi\rangle\langle\psi|$, then $F = \langle\psi|\sigma|\psi\rangle$
  - F is invariant under unitary transformations on either state.
  - For p_1, p_2 s.t. $p_1 + p_2 = 1$, $F(\rho, p_1\sigma + p_2\tau) \geq p_1 F(\rho, \sigma) + p_2 F(\rho, \tau)$
  - $F(\rho \otimes \sigma, \tau \otimes \nu) = F(\rho, \tau) F(\sigma, \nu)$

Fidelities for density matrices

- The fidelity between two density matrices ρ and σ can be defined as $F(\rho,\sigma) = \mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$

A sketch of the proof [1]:

- Write $\rho = \sum_i r_i |r_i\rangle\langle r_i|, \sigma = \sum_j s_j |s_j\rangle\langle s_j|$ as states on a Hilbert space H
- Purify these states with a second set of Hilbert spaces $H' = H$, get states

$$|\psi_\rho\rangle = \sum_i \sqrt{r_i}|r_i\rangle \otimes |r_i'\rangle, |\phi_\sigma\rangle = \sum_j \sqrt{s_j}\,|s_j\rangle \otimes |s_j'\rangle$$

- $F(|\psi_\rho\rangle\langle\psi_\rho|, |\phi_\sigma\rangle\langle\phi_\sigma|) = |\langle\psi_\rho|\phi_\sigma\rangle|^2 = \mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma}U)$ where U is unitary
- Now, because $\mathrm{Tr}(AU) \leq \mathrm{Tr}(|A|) = \mathrm{Tr}\left(\sqrt{A^\dagger A}\right) = \mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$
- Alternatives exist! [2]
- If one of these states is pure ($|\psi_\rho\rangle$), then we can write $F = \langle\psi_\rho|\sigma|\psi_\rho\rangle$

[1] https://entangledphysics.com/2019/06/24/quantum-fidelity-or-how-to-compare-quantum-states/
[2] https://arxiv.org/pdf/1810.08034.pdf

Maximizing entanglement: von Neumann entropy

- To maximize entanglement, we must first quantify entanglement.
- We can look at the von Neumann entropy (degree of mixing) of the constituent parts living in Hilbert spaces H_1 and H_2 [1]
  - $S = -\mathrm{Tr}\{\rho \log_2(\rho)\} = \sum_i \lambda_i \log_2(\lambda_i)$
  - $E = S_1$ or S_2 with $\rho_1 = \mathrm{Tr}_2\{\rho\}$, $\rho_2 = \mathrm{Tr}_1\{\rho\}$

Maximizing entanglement: concurrence

- For a pure state, define the concurrence as

$$C(\rho) = \sqrt{2[1 - \mathrm{Tr}\{\rho_A^2\}]}$$

- For a general 2-qubit state, we can define the concurrence [1]

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where the λ_i are the eigenvalues of the matrix

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$$

and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

The concurrence:

- Ranges from zero to one
- Is monotonically increasing with E [2]
- Doesn't generalize well to higher dimensions
- Is extremely nonlinear (as is E!)

[1] https://arxiv.org/pdf/quant-ph/9703041.pdf
[2] https://arxiv.org/pdf/quant-ph/0504163.pdf

Functionals for unitary optimization

- How do you define something like the fidelity for unitary optimization?
- To define something similar to $\lambda_0(1 - |\langle\psi_d|\psi(T)\rangle|^2)$ for an obtained unitary U relative to a desired unitary U_d, write

$$J_U(U, U_D) = \lambda_U \left(1 - \mathrm{Tr}\left\{U_D^\dagger U\right\}/d\right)$$

where d is the dimension of the system.

Practical quantum optimal control, a primer

When setting up a quantum optimal control problem, you need to do the following:

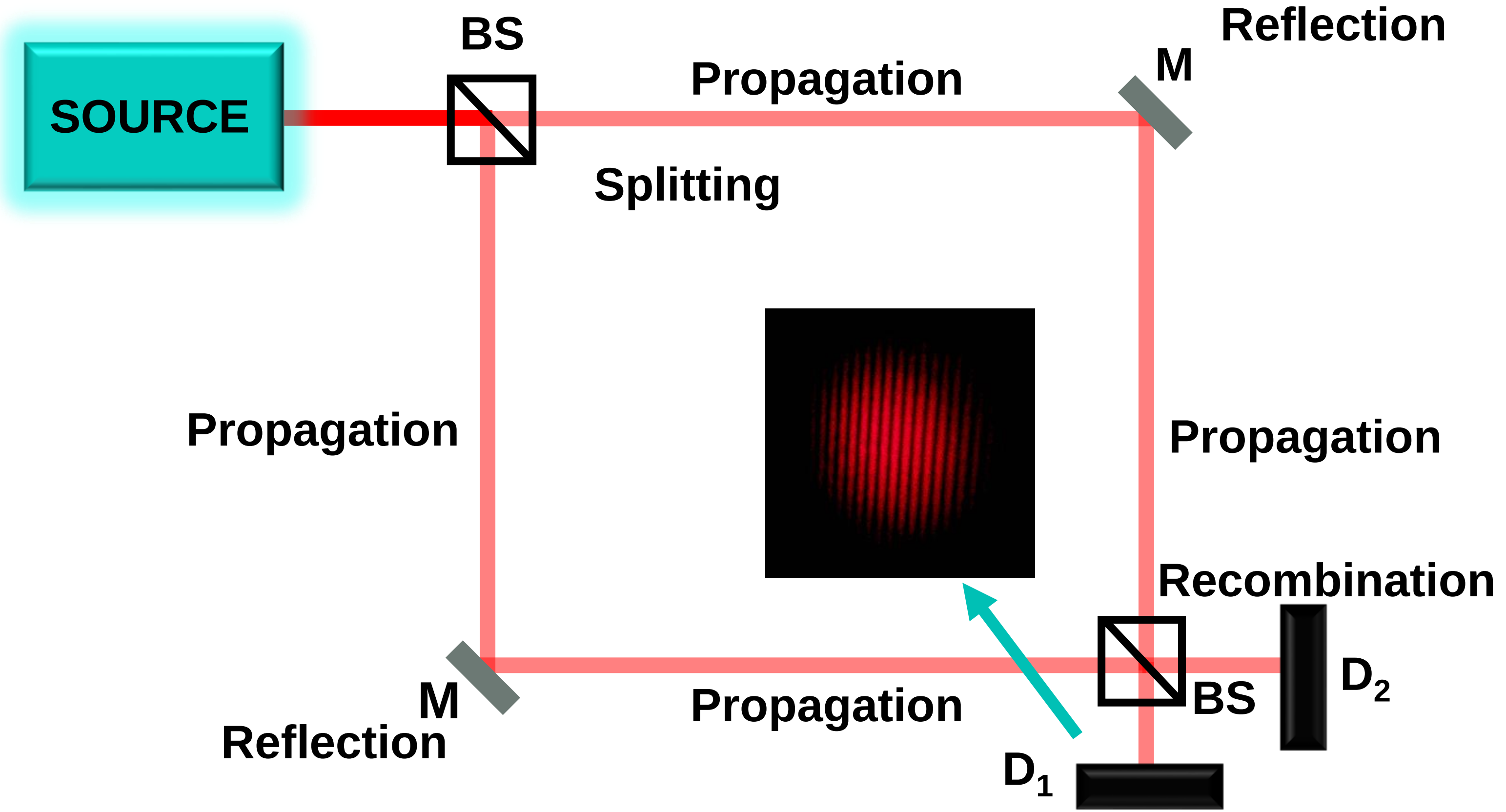
- Define your problem. Is it a state transfer problem or a unitary optimization problem?
- Determine what your controls are. That is, what knobs (real or virtual) can you turn to influence your system? What are their limits?
- Understand what information you can get out of your system. What can you measure?
- What algorithm will you use?
- Is there anything you can make use of that will simplify your problem?

A case study: sensing inertial motion with a shaken lattice atom interferometer

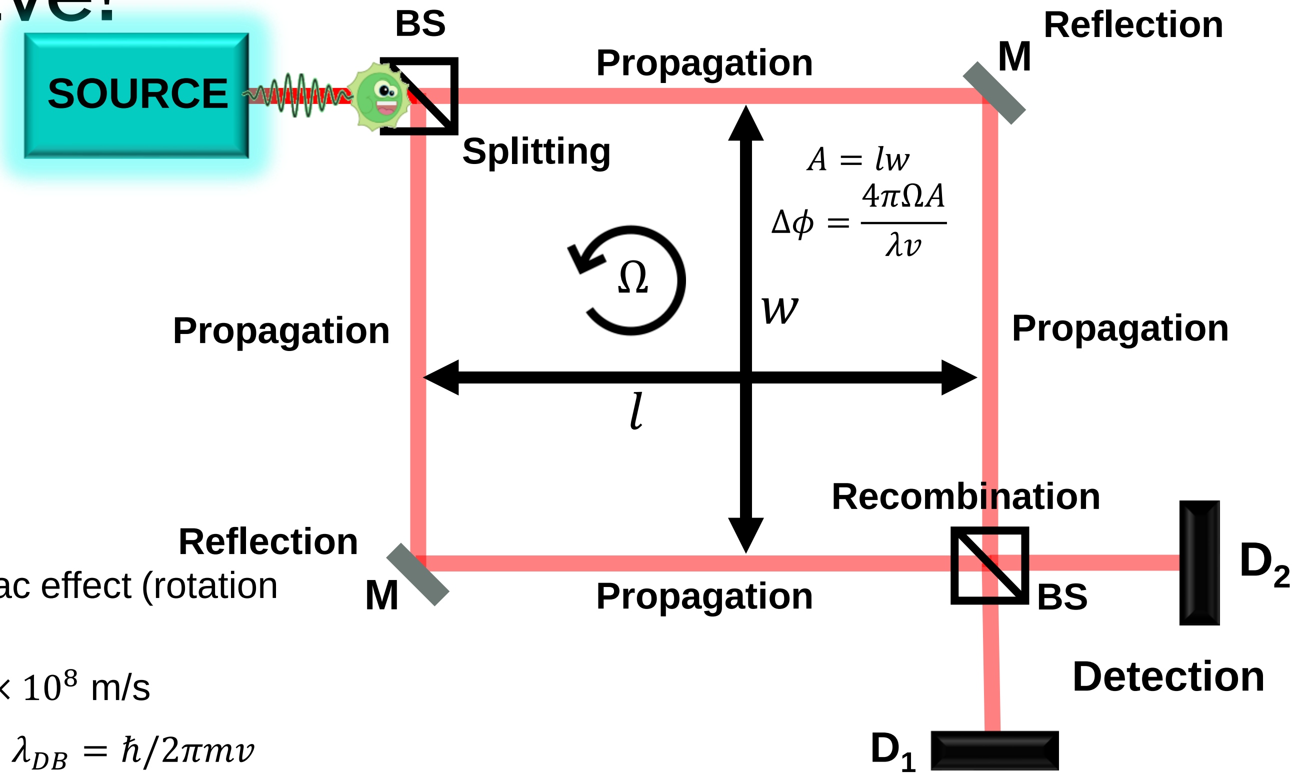
First, what is quantum sensing?

- Quantum sensing: using an inherently quantum device to sense a quantity
- Atom interferometry is just one option for building a quantum sensor (typically measures motion: acceleration/rotation)
- Cold atom systems have been used to sense magnetic fields via the Faraday effect
- We’re not just limited to atoms: NV centers make excellent sensors, as do NMR systems
- Applications range from inertial navigation to medical imaging to tests of fundamental physics

Traditional interferometry: an overview



Atom interferometers can be more sensitive!

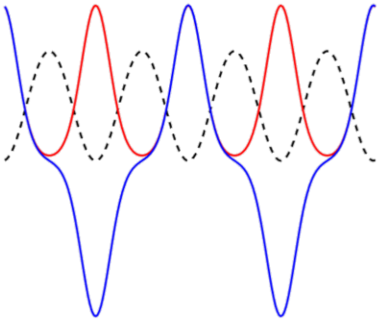


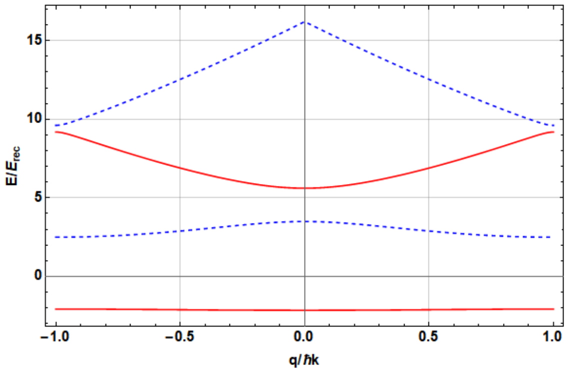
Consider the Sagnac effect (rotation sensing)

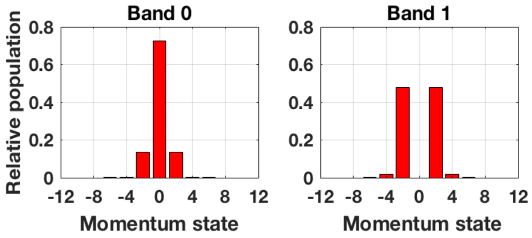
- Light: $v = c = 3 \times 10^8$ m/s
- Atoms (mass m): $\lambda_{DB} = \hbar/2\pi m v$
- $\frac{\Delta\phi_{atoms}}{\Delta\phi_{light}} = \frac{2\pi\lambda mc}{\hbar} \approx 10^{11}$

Reminder: how do we describe the atom wavefunction in a lattice?

- Bloch functions
  - Atoms delocalized in position, localized in momentum space
  - Gives rise to band structure within a Brillouin zone



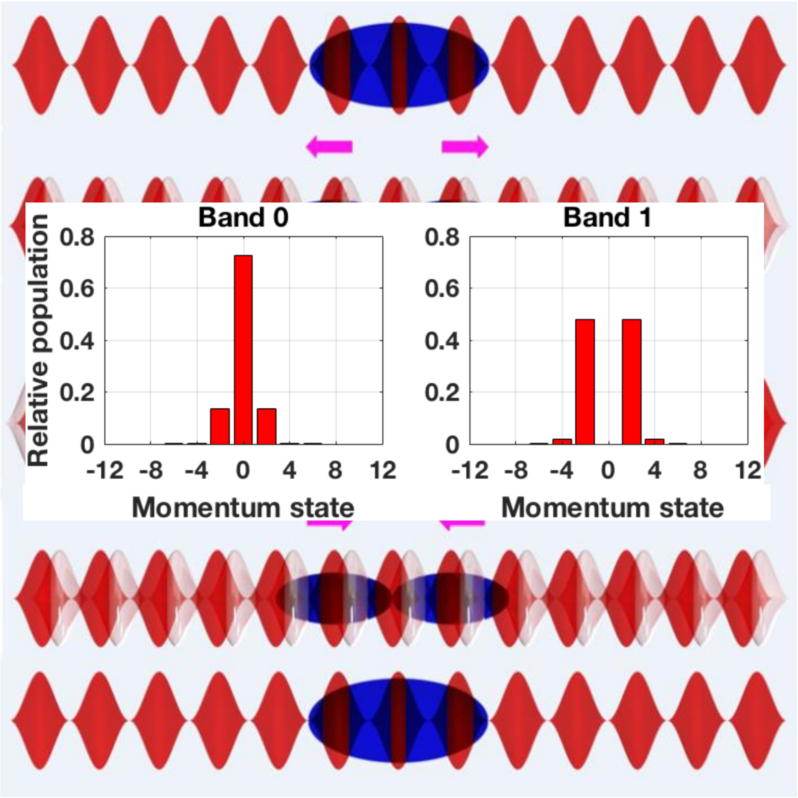




Building a shaken lattice interferometer, piece-by-piece

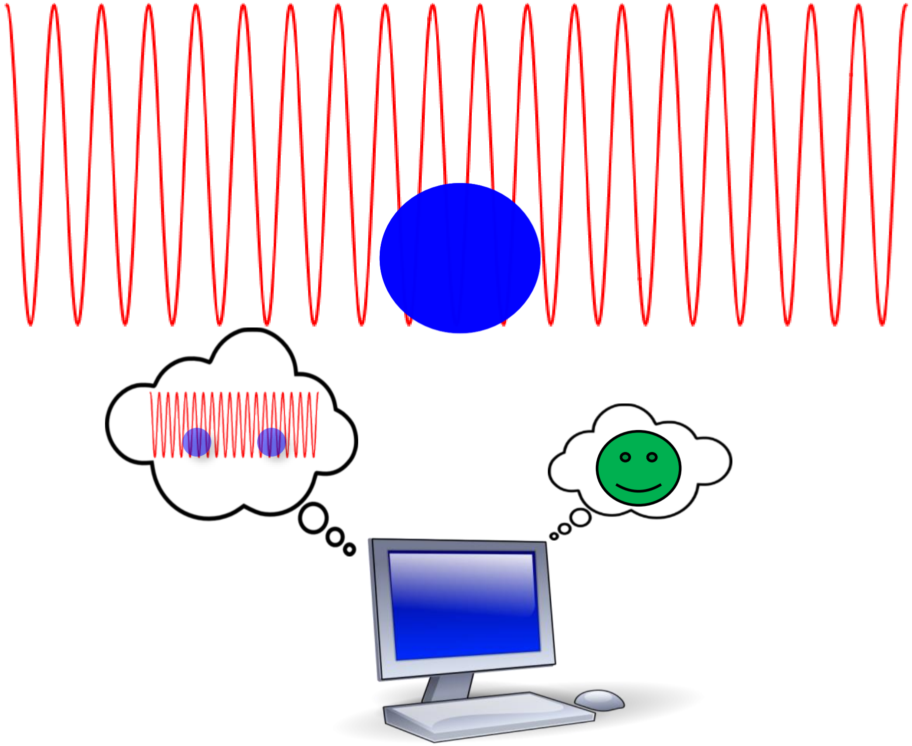
What we control!

- Trap atoms in an optical lattice potential:
  - $V(x,t) = V_0 \cos\left(2k_L x + \phi(t)\right)$
- Use the shaking to control the momentum state of the atoms
  - Atoms are delocalized in shallow lattice (Bloch states)
  - Model is limited to momenta quantized in units of $2\hbar k_L$
- Starting with atoms in the ground state of the lattice potential $|\psi_0\rangle$, we implement:
  - Splitting
  - Propagation
  - Reflection
  - Reverse propagation
  - Recombination back into ground state



How did we optimize the system?

- Start with an initial state and an objective function
  - e.g. convergence to a desired state
- By tailoring $\phi(t)$ to the desired response, can build an atom interferometer
- Use an optimization algorithm to “teach” the lattice to control the atoms
  - Work in Fourier basis: optimize a set of shaking frequencies
- Once the shaking function is known, it is fixed.
  - Can then calibrate the system’s response to a signal



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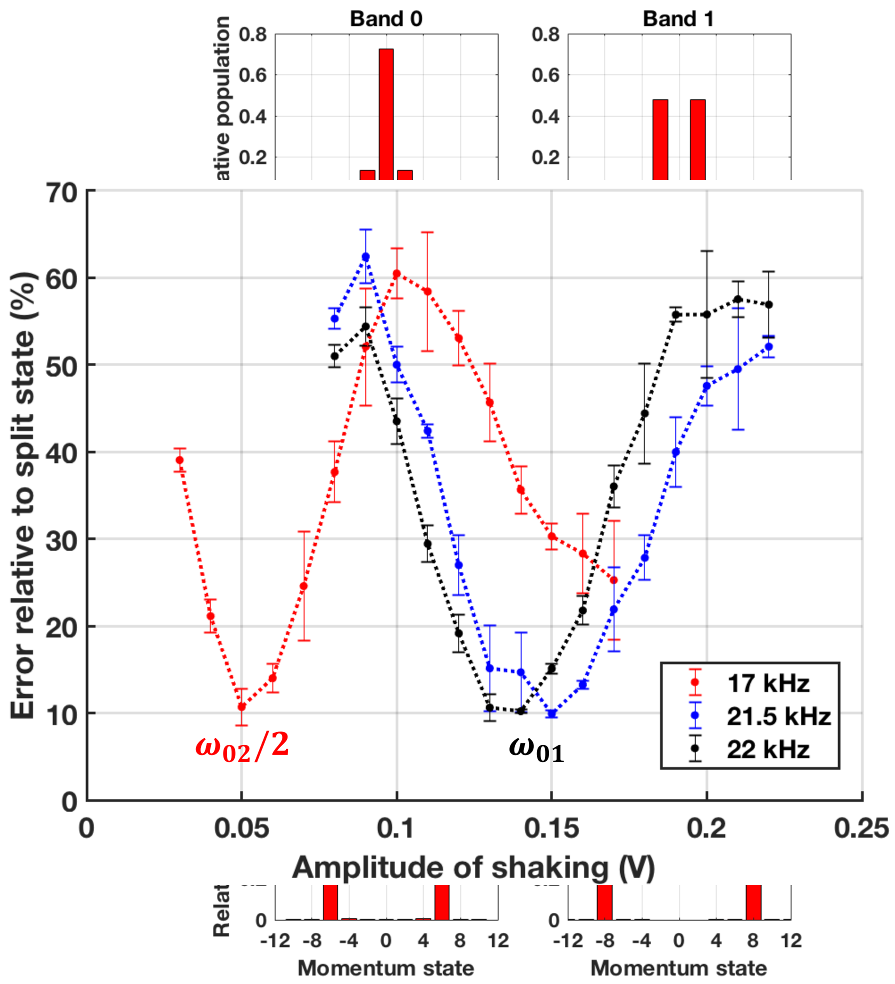
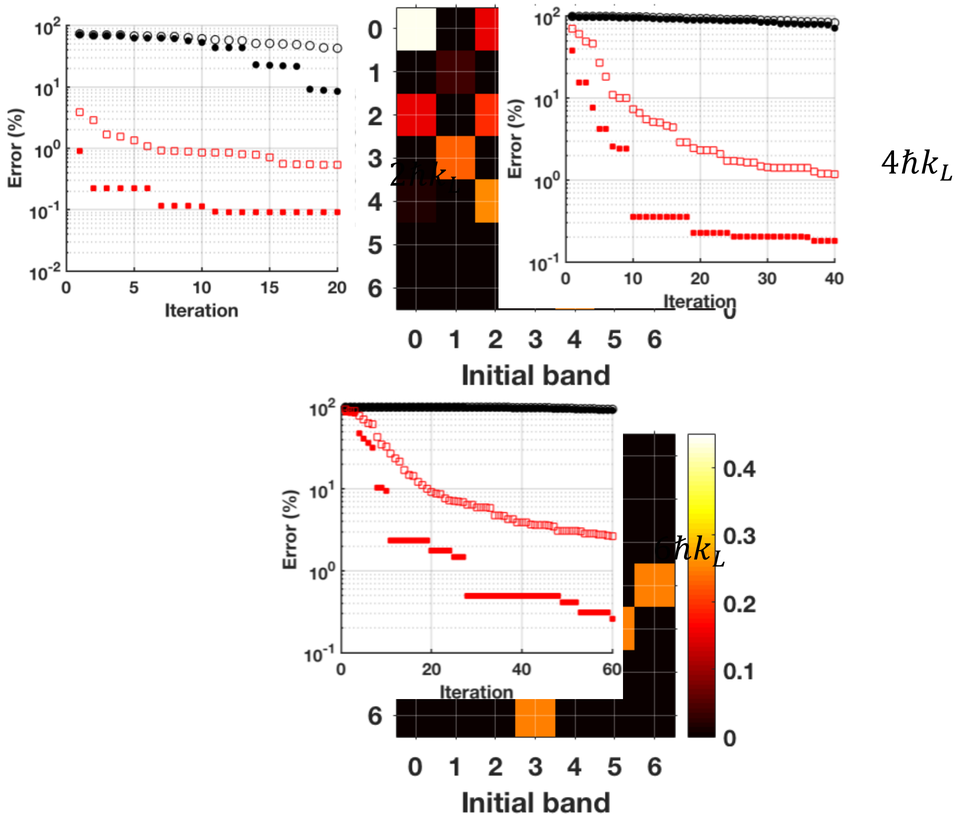
Being smarter about optimization

- Our initial simulations and experiments optimized a control of the form $u(t) = \sum_n c_n \cos(\omega_n t)$ for $n \approx 5$ random ω_n within a 100 kHz bandwidth
- However, the split states look like Bloch states!
- Idea: drive band-to-band transitions using phase modulation
  - Drive single (two)-photon transitions between opposite (same) parity states
  - $\Gamma \propto |\langle j|\sin(2k_L x)|i\rangle|^2$ or $|\langle j|\cos(2k_L x)|i\rangle|^2$
- We found that we could easily achieve first-order splitting ($\pm 2\hbar k_L$) with one frequency



Improved optimization with band transitions

- If we limit our optimization to just the band-to-band transitions with appreciable matrix element overlap, we can:
  - Optimize for higher-order splitting within 1%
  - Learn much more quickly
- Idea: run learning algorithm and optimize using only **selected** frequencies
- Learn to within 1% within 30 iterations
- By avoiding the “curse of high dimensionality”, we did substantially better.



What is the point? Broad takeaways

- The tools of atom trapping, cooling, and control allow for a variety of awesome experiments and technologies
- The luckiest of us can get away with open-loop control in an experiment (that is, a good theory model), but typically closed-loop control will be needed.
- Optimal control methods are varied and generally depend on the problem at hand.
- Avoid the curse of high-dimensionality if you can! Think: what physics can you apply to the problem?