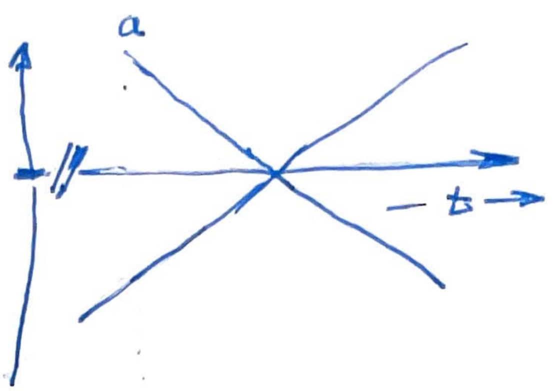
-I.1-

Landau-Zener problem



$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = H(t)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$H \equiv \hbar \begin{pmatrix} -\alpha t & g \\ g & \alpha t \end{pmatrix}$$ g is a frequency
α is a rate of change of frequency = chirp

dimensionless time : $\tau \equiv gt$ $\quad \varepsilon \equiv \frac{\alpha}{g^2}$

system of equations
$$\begin{cases} i\dot{a} = -\varepsilon\tau a + b \\ i\dot{b} = a + \varepsilon\tau b \end{cases} \quad (*)$$
dot indicates derivative with respect to τ

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decoupling of equations (standard approach)

$$i\ddot{a} = -\varepsilon a - \varepsilon\tau\dot{a} + \dot{b} \Rightarrow \ddot{a} = i\varepsilon a + \varepsilon\tau(i\dot{a}) - i\dot{b}$$

$$\ddot{a} = i\varepsilon a + \varepsilon\tau[-\varepsilon\tau a + b] - a - \varepsilon\tau b$$

$$(***) \qquad \ddot{a} + [(\varepsilon\tau)^2 + 1 - i\varepsilon]a = 0$$
solutions in terms of parabolic cylinder functions

harmonic oscillator with time-dependent frequency but complex coupling $a = a_r + ia_i$ (decomposition into real and imaginary parts)

$$\ddot{a}_r + [(\varepsilon\tau)^2 + 1]a_r = -\varepsilon a_i$$
$$\ddot{a}_i + [(\varepsilon\tau)^2 + 1]a_i = \varepsilon a_r$$
$$\omega(\tau) = \sqrt{1 + (\varepsilon\tau)^2}$$

two harmonic oscillators with time-dependent frequency and asymmetric coupling (signs on the right-hand side are different) non-reciprocal coupling;

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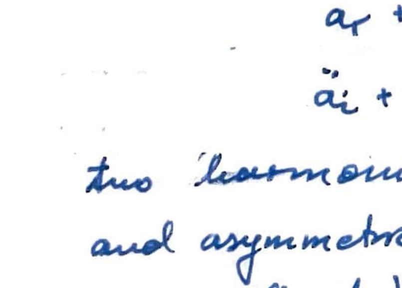
initial conditions: $a(\tau = -|\tau_0|) = 1$ (all population in excited state no phase)

$0 < \tau_0$:



$$a(\tau = -\tau_0) = 1 \quad (**)$$

unitary time evolution $|a|^2 + |b|^2 = 1$

$$|b(\tau = -\tau_0)| = 0$$

from Eq.(*) we find $\dot{a}(\tau = -\tau_0) = i\varepsilon\tau_0$

one-line derivations

$$\dot{a} = i\varepsilon\tau a - ib$$
$$\dot{b} = -ia - i\varepsilon\tau b$$

; ansatz: $a(\tau) \equiv e^{+i\varepsilon\tau^2/2}\,\tilde{a}(\tau)$

$$\dot{a} = i\varepsilon\tau a + e^{i\varepsilon\tau^2/2}\dot{\tilde{a}}$$

ansatz: $b = e^{-i\varepsilon\tau^2/2}\,\tilde{b}$

$$\dot{b} = -i\varepsilon\tau b + e^{-i\varepsilon\tau^2}\dot{\tilde{b}}$$

$$\dot{\tilde{a}} = -ie^{-i\varepsilon\tau^2/2}\,b$$

$$\boxed{\begin{array}{l} \dot{\tilde{a}} = -ie^{-i\varepsilon\tau^2}\,\tilde{b} \\ \hline \dot{\tilde{b}} = -ie^{i\varepsilon\tau^2}\,\tilde{a} \end{array}}$$

initial conditions: $a(\tau = -\tau_0) = 1$

$$\tilde{a}(\tau = -\tau_0) \equiv e^{-i\varepsilon\tau_0^2/2}$$

from the normalization $\} \Rightarrow |b(\tau = -\tau_0)| = 0$ $\quad b(\tau = -\tau_0) \overset{\text{choose phase}}{=} 0$

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formal solution:

$$\tilde{b}(\tau) = -i\int_{-\tau_0}^{\tau} d\tau' e^{i\varepsilon\tau'^2}\tilde{a}(\tau')$$

hence

$$\dot{\tilde{a}} = -e^{-i\varepsilon\tau^2}\int_{-\tau_0}^{\tau} d\tau' e^{i\varepsilon\tau'^2}\tilde{a}(\tau')$$

integro-differential equation

Markov approximation: $\tilde{a}(\tau') \cong \tilde{a}(\tau)$

$$\dot{\tilde{a}}_M = \underbrace{-e^{-i\varepsilon\tau^2}\int_{-\tau_0}^{\tau} d\tau' e^{i\varepsilon\tau'^2}}_{\equiv\, \eta_M}\tilde{a}_M(\tau) \equiv -\eta_M(\tau)\,\tilde{a}_M(\tau)$$

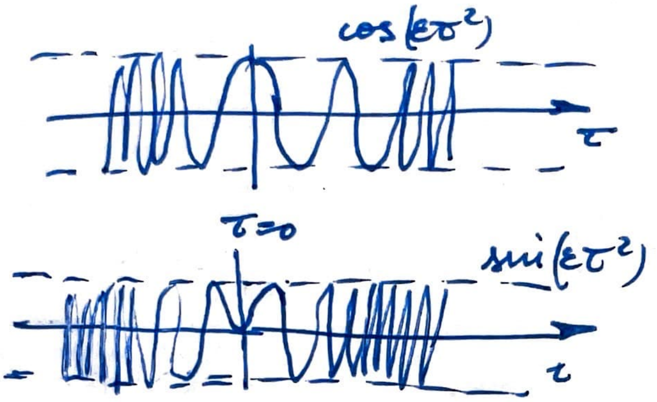
$$\tilde{a}_M(\tau) = e^{-i\varepsilon\tau_0^2/2}\exp\left[-\int_{-\tau_0}^{\tau} d\tau'\eta_M(\tau')\right]$$

$$a(\tau) \cong a_M(\tau) = e^{i\varepsilon(\tau^2-\tau_0^2)/2}\exp\left[-\int_{-\tau_0}^{\tau} d\tau'\eta_M(\tau')\right]$$

$$\eta_M(\tau) = e^{-i\varepsilon\tau^2}\int_{-\infty}^{\tau} d\tau' e^{i\varepsilon\tau'^2} = e^{-i\varepsilon\tau^2}\int_{0}^{\infty} d\bar\tau\, e^{i\varepsilon(\tau-\bar\tau)^2}$$

$\uparrow$ $\tau_0 \to \infty$ $\qquad$ $\uparrow$ $\bar\tau = \tau - \tau'$

$$= \int_0^\infty d\bar\tau\, e^{i\varepsilon[\bar\tau^2 - 2\tau\bar\tau]}$$



$S \equiv \bar\tau^2 - 2\tau\bar\tau$

$S' = 2(\bar\tau - \tau)$

integration variable is always positive; if τ is negative $S' > 0$ if τ is positive there is a point of stationary phase

$\bar\tau_s = \tau$

elementary approximation:

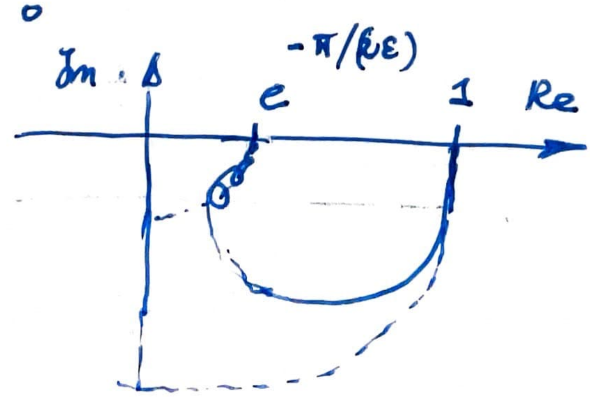
$$\eta_M(\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ e^{-i\varepsilon\tau^2}\underbrace{\int_{-\infty}^{\infty} d\tau' e^{i\varepsilon\tau'^2}}_{\sqrt{\frac{\pi}{-i\varepsilon}}} & \text{for } 0 < \tau \end{cases}$$

$$\eta_M(\tau) \simeq \begin{cases} 0 & \text{for } \tau < 0 \\ \sqrt{\frac{\pi}{-i\varepsilon}}\, e^{-i\varepsilon\tau^2} & \text{for } 0 < \tau \end{cases}$$

$\uparrow$ origin of the Stückelberg oscillations

$$\int_{-\infty}^{\infty} d\tau\, \eta_M(\tau) = \int_0^\infty d\tau \sqrt{\frac{\pi}{-i\varepsilon}}\, e^{-i\varepsilon\tau^2} = \frac{1}{2}\sqrt{\frac{\pi}{(-i\varepsilon)}}\sqrt{\frac{\pi}{+i\varepsilon}} = \frac{\pi}{2\varepsilon}$$

$\exp\left[-\int_{-\tau_0}^{T} d\tau' \eta_M(\tau')\right]$



second derivation:

$$\ddot{\tilde{a}} = (-i2\varepsilon\tau)\dot{\tilde{a}} - i e^{-i\varepsilon\tau^2}\dot{\tilde{b}} =$$

$$= (-i2\varepsilon\tau)\dot{\tilde{a}} - i e^{-i\varepsilon\tau^2}(-i e^{i\varepsilon\tau^2} a)$$

$$= -i2\varepsilon\tau\dot{\tilde{a}} - a;$$

$$\ddot{\tilde{a}} + i2\varepsilon\tau\dot{\tilde{a}} + \tilde{a} = 0$$

neglect $\ddot{\tilde{a}} \cong 0$

$$i2\varepsilon\tau\dot{\tilde{a}} + \tilde{a} = 0$$

singularity at $\tau = 0$ add a real positive number δ to avoid the singularity $\delta\dot{\tilde{a}}$

$$[\delta + i(2\varepsilon\tau)]\dot{\tilde{a}} + \tilde{a} = 0$$

$$\dot{\tilde{a}} = -\frac{1}{\delta + i(2\varepsilon\tau)}\tilde{a} = -\frac{\delta - i(2\varepsilon\tau)}{\delta^2 + (2\varepsilon\tau)^2}\tilde{a}$$

$$\tilde{a}(\tau) = e^{-i\varepsilon\tau_0^2/2}\exp\left[-\int_{-\tau_0}^{\tau} d\tau' \frac{\delta}{\delta^2 + (2\varepsilon\tau')^2} + i\int_{-\tau_0}^{\tau} d\tau' \frac{2\varepsilon\tau'}{\delta^2 + (2\varepsilon\tau')^2}\right]$$

$\delta \to 0$

$$\downarrow = e^{-i\varepsilon\tau_0^2/2}\exp\left[-\pi\int_{-\tau_0}^{\tau} d\tau'\,\delta(2\varepsilon\tau') + i\frac{1}{2\varepsilon}\int_{-\tau_0}^{\tau} d\tau' \frac{1}{\tau'}\right]$$

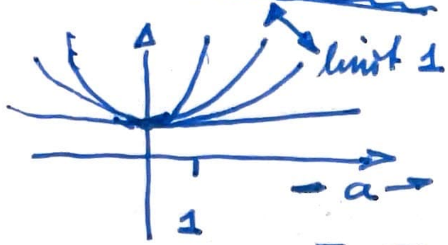
Cauchy principal part

$$= e^{-i\varepsilon\tau_0^2/2}\exp\left[-\frac{\pi}{2\varepsilon}\Theta(\tau) + i\frac{1}{2\varepsilon}\int_{-\tau_0}^{\tau} d\tau' \frac{1}{\tau'}\right]$$

jump at $\tau = 0$ logarithmic phase

$$a(\tau = \tau_0) = e^{-\frac{\pi}{2\varepsilon}}$$

$a_r(\tau = -\tau_0) = 1$ $\qquad$ $a_i(\tau = -\tau_0) = 0$



potential of oscillator
coupling is only important for $\tau \cong 0$

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$$\lim_{\tau_0 \to \infty} a(\tau = \tau_0) = e^{-\pi/(2\varepsilon)}$$ (asymptotic limit of parabolic cylinder functions)

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Different approach: Ricatti equation

ansatz: $$a(\tau) = e^{-i\varepsilon(\tau_0^2 - \tau^2)/2} \exp\left[-\int_{-\tau_0}^{\tau} d\tau' \eta(\tau')\right]$$

ensure the initial condition (**)

(?) $\dot{a} = i\varepsilon\tau a - \eta a$ $\qquad$ $\ddot{a} = i\varepsilon a + i\varepsilon\tau\dot{a} - \dot{\eta}a - \eta\dot{a}$

$$= i\varepsilon a + i\varepsilon\tau(i\varepsilon\tau a - \eta a) - \dot{\eta}a - \eta(i\varepsilon\tau a - \eta a)$$

$$= \left[\eta^2 - 2i\varepsilon\tau\eta - \dot{\eta} - (\varepsilon\tau)^2 + i\varepsilon\right] a =$$

Eq.(***)

$$= \left[-(\varepsilon\tau)^2 - 1 + i\varepsilon\right] a;$$

$$(\eta^2 - 2i\varepsilon\tau\eta - \dot{\eta} + 1)\, a = 0$$

$$\boxed{\eta^2 - 2i\varepsilon\tau\eta - \dot{\eta} + 1 = 0}$$ complex-valued Ricatti equation

From Eq(*) we find $\quad b = i\dot{a} + \varepsilon\tau a$ and with Eq(?)

$$b = i(i\varepsilon\tau a - \eta a) + \varepsilon\tau a$$

$$\boxed{b = -i\eta a} \quad (!)$$

due to $b(\tau = -\tau_0) = 0$ and $a(\tau = -\tau_0) = 1$ we find

$$\eta(\tau = -\tau_0) = 0$$

$\eta = \eta_r + i\eta_i$ $\qquad$ decomposition of η into real and imaginary parts

$$\varphi(\tau) = \varepsilon(\tau^2 - \tau_0^2)/2 - \int_{-\tau_0}^{\tau} d\tau' \eta_i(\tau')$$

$$a(\tau) = e^{i\varphi(\tau)} \exp\left[-\int_{-\tau_0}^{\tau} d\tau' \, \eta_r(\tau')\right] = e^{i\varphi(\tau)} A(\tau)$$

$$b(\tau) = e^{i\varphi(\tau)} e^{i\psi(\tau)} |\eta| \, A$$

$$-i\eta = |\eta| e^{i\psi} \qquad \psi = \arg\eta - \pi/2$$

normalisation condition : $1 = |a|^2 + |b|^2 = A^2 + |\eta|^2 A^2$

$$|\eta|^2 = \frac{1-A^2}{A^2} \qquad |\eta| = \sqrt{\frac{1-A^2}{A^2}}$$

$$\boxed{\begin{aligned} a(\tau) &= e^{i\varphi(\tau)} A(\tau) \\ b(\tau) &= e^{i\varphi(\tau)} \sqrt{1-A^2(\tau)}\, e^{i\psi(\tau)} \end{aligned}}$$

- a and b share the overall phase φ ;
- A is determined by integral of η_r (real part of η)
- b is determined by phase ψ governed by phase of η (not real or imaginary part but the combination)

approximations : $\dot{b} = -i\varepsilon\tau b - ia$

formal solution with the initial condition $b(\tau = -\tau_0) = 0$

$$b(\tau) = -i \, e^{-i\varepsilon\tau^2/2} \int_{-\tau_0}^{\tau} d\tau' e^{i\varepsilon\tau'^2/2} a(\tau')$$

very different from Eq (*)

substitute into Eq. (*):

$$i\dot{a} = -\varepsilon\tau a - i\, e^{-i\varepsilon\tau^2/2} \int_{-\tau_0}^{\tau} d\tau' e^{i\varepsilon\tau'^2/2} a(\tau')$$

linear integral-differential equation of first order

ansatz: $a(\tau) = e^{i\varepsilon(\tau^2-\tau_0^2)/2} \tilde{a}(\tau) \Rightarrow \dot{a} = i\varepsilon\tau a + e^{i\varepsilon(\tau^2-\tau_0^2)/2} \dot{\tilde{a}}$

$$\to i e^{i\varepsilon(\tau^2-\tau_0^2)/2} \dot{\tilde{a}} = -i e^{-i\varepsilon\tau^2/2} e^{-i\varepsilon\tau_0^2/2} \int_{-\tau_0}^{\tau} d\tau' \tilde{a}(\tau')$$

$$\boxed{\dot{\tilde{a}} = -e^{-i\varepsilon\tau^2} \int_{-\tau_0}^{\tau} d\tau' \tilde{a}(\tau')}$$

approximations of <u>Ricatti equation</u> (neglect non-linearity)

$$\dot{\eta} = -2i\varepsilon\tau\eta + 1 + \eta^2$$

formal solution subjected to the initial condition, Eq.

$$\eta(\tau) = e^{-i\varepsilon\tau^2}\int_{-\tau_0}^{\tau} d\tau\, e^{+i\varepsilon\tau^2}(1+\eta^2)$$

$$\eta(\tau) = e^{-i\varepsilon\tau^2}\int_{-\tau_0}^{\tau} d\tau\, e^{+i\varepsilon\tau^2} + e^{-i\varepsilon\tau^2}\int_{-\tau_0}^{\tau} d\tau'\, e^{i\varepsilon\tau'^2}\,\eta^2(\tau')$$

non-linear integral equation:

$$\eta_H(\tau) \doteq e^{-i\varepsilon\tau^2}\int_{-\tau_0}^{\tau} d\tau\, e^{i\varepsilon\tau^2}$$

argument of exponential in A:

$$\int_{-\tau_0}^{\tau_0} d\tau'\,\eta_H(\tau') = \int_{-\tau_0}^{\tau_0} d\tau'\, e^{-i\varepsilon\tau'^2}\int_{-\tau_0}^{\tau'} d\tau''\, e^{i\varepsilon\tau''^2} =$$

$$= \underbrace{\int_{-\tau_0}^{\tau_0} d\tau' \ldots \int_{-\tau_0}^{0} d\tau''}_{\text{symmetric}} + \underbrace{\int_{-\tau_0}^{\tau_0} d\tau' \ldots \underbrace{\int_{0}^{\tau'} d\tau''}_{\text{symmetric}}}_{\substack{\text{anti-symmetric}\\ =0}}$$

$$= \frac{1}{2}\left|\int_{-\tau_0}^{\tau_0} d\tau'\, e^{-i\varepsilon\tau'^2}\right|^2$$

$$\tau_0 \to \infty$$

$$= \frac{1}{2}\,\frac{\pi}{\varepsilon} = \frac{\pi}{2\varepsilon};$$

$$\int_{-\infty}^{\infty} dx\, e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}\,; \quad \int_{-\infty}^{\infty} d\tau\, e^{-i\varepsilon\tau^2} = \int_{-\infty}^{\infty} d\tau\, e^{-(i\varepsilon)\tau^2} = \sqrt{\frac{\pi}{i\varepsilon}}$$

approximations for $-\tau_0 < \tau$:

neglect $\eta^2 = 0$ and $\dot{\eta} \hat{=} 0$ and iterate the equation

initial condition

$2i\varepsilon\tau\eta \hat{=} 1 \Rightarrow \eta = \frac{1}{2i\varepsilon\tau} = -\frac{i}{2\varepsilon}\frac{1}{\tau}$ purely imaginary

$\int_{-\tau_0}^{\tau} d\tau' \eta(\tau') = -\frac{i}{2\varepsilon} \ln \frac{|\tau|}{\tau_0}$ logarithmic phase

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differential equation for η_M. $\dot{\eta}_M = -2i\varepsilon\tau\eta_M + 1$

References:

- E. Glasbrenner and W. P. Schleich, J. Phys. B 56, 104001 (2023)
- E. Glasbrenner, Y. Gerdes, S. Varro and W. P. Schleich, in, Festschrift for V. Dodonov, Chapter 17 (2024)
- E. Glasbrenner, Y. Gerdes, S. Varro and W. P. Schleich to be published